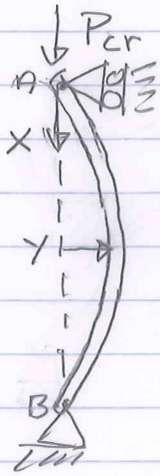
35. Buckling of an elastic column loaded at the centroid of the cross-section
Beam differential equation applies; constant EI; column is perfectly straight; axial force P and support axial force are applied through the centroid.

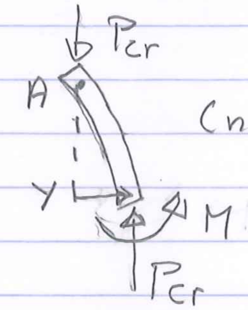
P_{cr} is the value of P that can hold the column in a slightly deformed position.



column length l

Column is pinned top and bottom. Note that the horizontal support reactions are zero (no shear in deformed column).
To apply the beam differential equation, the lateral deflection y is assumed to be small.

Consider a FBD of an upper part of the deformed column.



(no shear force)

$$\Sigma M_A = 0 = M + P_{cr} \cdot y$$
$$\Rightarrow M = -P_{cr} \cdot y$$

Apply beam equation:

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{-P_{cr} \cdot y}{EI}$$

$$\frac{d^2y}{dx^2} + \frac{P_{cr}\, y}{EI} = 0$$ This is an eigenvalue problem.

$$y = a \cdot \sin\sqrt{\frac{P_{cr}}{EI}}\,x + b \cdot \cos\sqrt{\frac{P_{cr}}{EI}}\,x$$, where a and b

are found from the boundary conditions.

$y(x=0) = 0 \Rightarrow b = 0$

$y(x=\ell) = 0 \Rightarrow a \cdot \sin\sqrt{\frac{P_{cr}}{EI}}\,\ell = 0$

(Note: $M(x=0)=0$ and $M(x=\ell)=0$ are automatically satisfied)

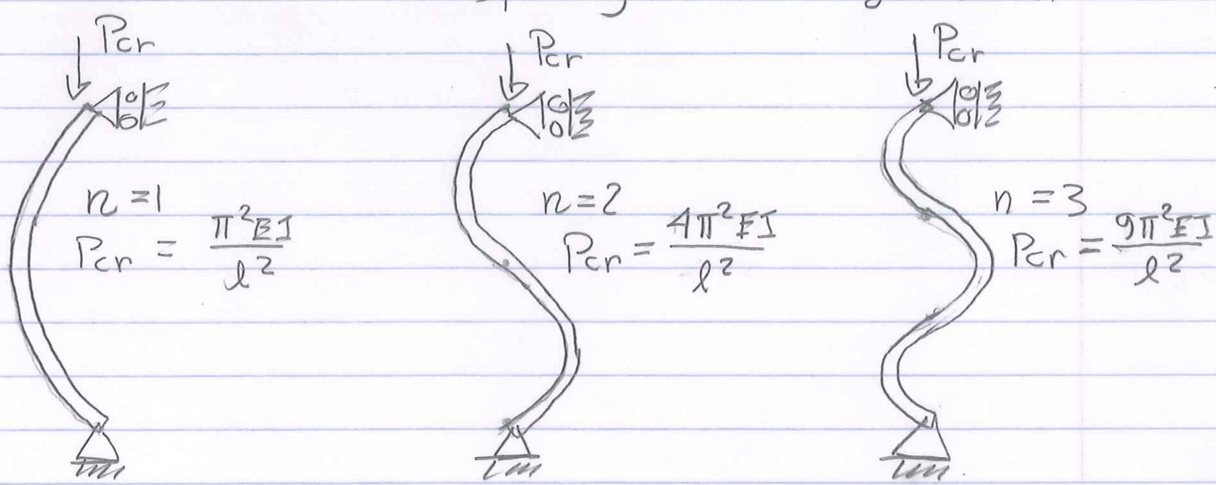
$a=0$ gives a trivial solution, so we seek values of P_{cr} (the eigenvalues) for which

$$\sin\sqrt{\frac{P_{cr}}{EI}}\,\ell = 0$$

Thus, $\sqrt{\frac{P_{cr}}{EI}}\,\ell = n\pi$

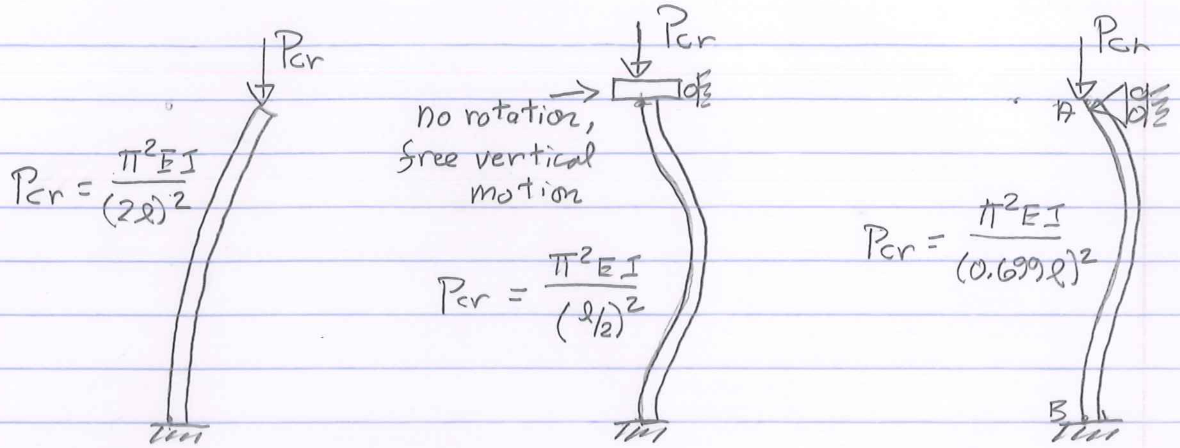
so, $P_{cr} = \frac{n^2\pi^2 EI}{\ell^2}$ for $n = 1, 2, 3, \ldots$ This is the Euler buckling formula. The buckled shapes

$y = A\sin\sqrt{\frac{P_{cr}}{EI}}\,\ell$ are the eigenfunctions, one corresponding to each eigenvalue.

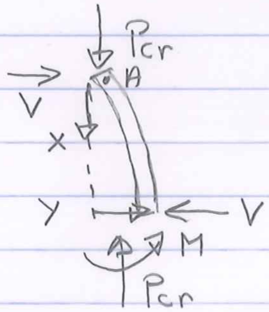


Note that the $n>1$ cases can not occur physically without intermediate lateral supports, since the $n=1$ buckling would occur first.

Different boundary conditions will produce different values of P_{cr}. Some examples are shown below for the lowest buckling load.



The two cases on the left are derived as the previous example, just different boundary conditions. The case on the right is different because shear force is present in the deformed column. For this case consider a FBD of the upper part.



$$\Sigma M_A = 0 = M + P_{cr}\cdot y - V\cdot x$$

$$\Rightarrow M = -P_{cr}\cdot y + V\cdot x$$

Apply beam equation:

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{-P_{cr}\cdot y}{EI} + \frac{V\cdot x}{EI}$$

$$\frac{d^2y}{dx^2} + \frac{P_{cr}\cdot y}{EI} = \frac{V\cdot x}{EI}$$

$$y = a\cdot \sin\sqrt{\frac{P_{cr}}{EI}}\,x + b\cdot\cos\sqrt{\frac{P_{cr}}{EI}}\,x + \frac{V\cdot x}{P_{cr}}$$

$y(x=0) = 0 \Rightarrow b = 0$; $y(x=l) = 0 \Rightarrow a\cdot\sin\sqrt{\frac{P_{cr}}{EI}}\,l + \frac{V\cdot l}{P_{cr}} = 0$ (1)

$\left.\frac{dy}{dx}\right|_{x=l} = 0 \Rightarrow a\cdot\sqrt{\frac{P_{cr}}{EI}}\cdot\cos\sqrt{\frac{P_{cr}}{EI}}\,l + \frac{V}{P_{cr}} = 0$ (2) (Note: $M(x=0)=0$ is auto. satisfied)

From (1) and (2): $a\cdot\sin\sqrt{\frac{P_{cr}}{EI}}\,l - l\,a\cdot\sqrt{\frac{P_{cr}}{EI}}\cdot\cos\sqrt{\frac{P_{cr}}{EI}}\,l = 0$

simplify: $\tan\sqrt{\frac{P_{cr}}{EI}}\,l = \sqrt{\frac{P_{cr}}{EI}}\,l \Rightarrow P_{cr} = \frac{\pi^2 EI}{(0.699\,l)^2}$

Note that all of the formulas for P_{cr} can be written as

$$P_{cr} = \frac{\pi^2 EI}{(k\ell)^2}$$

where k is the effective length factor. $k\ell$ is the partial length of the column between points of inflection (where $M = 0$).

$k = 1$ for pins at both ends

$= 2$ for cantilever

$= \frac{1}{2}$ for both ends fixed against rotation

$= 0.6999$ for the pinned-fixed case